# Exercise 10

Use the *successive approximations method* to solve the following Volterra integral equations:

$$u(x) = 1 - 2\sinh x + \int_0^x (x - t + 2)u(t)\,dt$$

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### Solution

The successive approximations method, also known as the method of Picard iteration, will be used to solve the integral equation. Consider the iteration scheme,

$$\begin{aligned} u_{n+1}(x) &= 1 - 2\sinh x + \int_0^x (x - t + 2)u_n(t)\,dt, \quad n \geq 0 \\ &= 1 - 2\sinh x + \int_0^x (x - t)u_n(t)\,dt + 2\int_0^x u_n(t)\,dt \\ &= 1 - 2\sinh x + \int_0^x \int_0^r u_n(t)\,dt\,dr + 2\int_0^x u_n(t)\,dt, \end{aligned}$$

choosing $u_0(x) = 0$. Then

$$\begin{aligned} u_1(x) &= 1 - 2\sinh x + \int_0^x (x - t + 2)u_0(t)\,dt = 1 \\ u_2(x) &= 1 - 2\sinh x + \int_0^x (x - t + 2)u_1(t)\,dt = 1 + 3x \\ u_3(x) &= 1 - 2\sinh x + \int_0^x (x - t + 2)u_2(t)\,dt = 1 + 3x + \frac{9}{2}x^2 \\ u_4(x) &= 1 - 2\sinh x + \int_0^x (x - t + 2)u_3(t)\,dt = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 \\ &\vdots, \end{aligned}$$

and the general formula for $u_{n+1}(x)$ is

$$u_{n+1}(x) = \sum_{k=0}^{n} \frac{(3x)^k}{k!}.$$

Take the limit as $n \to \infty$ to determine $u(x)$.

$$\begin{aligned} \lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(3x)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(3x)^k}{k!} \\ &= e^{3x} \end{aligned}$$

Therefore, $u(x) = e^{3x}$.